Nonperturbative mean-field theory for minimum enstrophy relaxation

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The dual cascade of enstrophy and energy in quasi-two-dimensional turbulence strongly suggests that a viscous but otherwise potential vorticity (PV) conserving system decays selectively toward a state of minimum potential enstrophy. We derive a *nonperturbative* mean field theory for the dynamics of minimum enstrophy relaxation by constructing an expression for PV flux during the relaxation process. The theory is used to elucidate the structure of anisotropic flows emerging from the selective decay process. This structural analysis of PV flux is based on the requirements that the mean flux of PV dissipates total potential enstrophy but conserves total fluid kinetic energy. Our results show that the structure of PV flux has the form of a sum of a positive definite hyperviscous and a negative or positive viscous transport of PV. Transport parameters depend on zonal flow and turbulence intensity. Turbulence spreading is shown to be related to PV mixing via the link of turbulence energy flux to PV flux. In the relaxed state, the ratio of the PV gradient to zonal flow velocity is homogenized. This homogenized quantity sets a constraint on the amplitudes of PV and zonal flow in the relaxed state. A characteristic scale is defined by the homogenized quantity and is related to a variant of the Rhines scale. This relaxation model predicts a relaxed state with a structure which is consistent with PV staircases, namely, the proportionality between mean PV gradient and zonal flow strength.

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I. INTRODUCTION

The formation of large-scale shearing structures due to momentum transport, i.e., zonal flow formation, is a common feature of both geostrophic fluids and magnetically confined plasmas (e.g., Refs. [1–5]). In this work, we study the dynamics of relaxation leading to structure formation. The relaxed state of a high Reynolds number, turbulent, two-dimensional (2D) fluid is thought to be one of minimum potential enstrophy, for given conserved kinetic energy. This hypothesis constitutes the minimum enstrophy principle of Bretherton and Haidvogel [6]. Their variational argument is based on the concept of selective decay, which is in turn based on the dual cascade in 2D turbulence. In 2D turbulence, kinetic energy inverse cascades to large, weakly dissipated spatial scales, whereas enstrophy forward cascades to small spatial scales, and there it is viscously damped. In the presence of weak dissipation, total kinetic energy is thus approximately conserved relative to total enstrophy, which is dissipated. Thus, the system evolves toward a state of a minimum enstrophy. Interestingly, the theory does not specify the minimum enstrophy actually achieved in the relaxed state. The theory predicts the structure of the flow in the end state; however, it gives no insight into the all-important question of how the mean profiles evolve during the relaxation process. Here we discuss the *dynamics* of minimum enstrophy relaxation, which leads to zonal flow formation. In particular, since inhomogeneous potential vorticity (PV) mixing is the fundamental mechanism of zonal flow formation, we ask *what form must the mean field PV flux have so as to dissipate enstrophy while conserving energy*?

The reason mixing of PV is the key element of zonal flow formation is that PV conservation is the fundamental freezing-in law constraint on zonal flow generation by inhomogeneous PV mixing. Note that since zonal flows are elongated, asymmetric vortex modes, translation symmetry in the direction of the flow and inhomogeneity across the direction of the flow are essential elements in zonal flow formation. The importance of PV mixing to the zonal flow problem is clearly seen via the Taylor identity, which states that the cross-flow flux of PV equals the along-flow component of the Reynolds force, which drives the flow. Most of the theoretical calculations of PV flux are modulational stability analyses using weak turbulence theory (e.g., Refs. [3,7,8]). These types of analyses are, however, valid only in the initial stage of zonal flow formation. Therefore, there is a need to develop a mean field theory based on general, structural principles, and not limited by perturbative methods. To obtain the general form of the PV flux, the selective decay hypothesis is exploited. In this paper, we show that the structure of the PV flux which dissipates enstrophy in mean field theory is $\Gamma_q = \langle v_x \rangle^{-1} \nabla [\mu \nabla (\nabla \langle q \rangle / \langle v_x \rangle)]$. In other words, PV flux is not given by a simple Fick's law but has a complex form involving viscosity and hyperviscosity, with flow-dependent transport coefficients. *In the relaxed state, the ratio between the local PV gradient and zonal flow is homogenized.* Interestingly, this proportionality relationship between PV gradient and zonal flow is observed in PV staircases.

We note that selective decay is a hypothesis based on the observation of the dual cascade in 2D turbulence and is not rigorously derived from first physical principles. There are relaxed states derived from more fundamental principles, namely, statistical equilibrium states and stable stationary states (see Refs. [9–11]). Even though the minimum enstrophy principle is not a first principle physical theory, it is a plausible and demonstrably useful guide, which gives us predictions of the structure of PV and flows, and the enstrophy level in the relaxed state. Although the validity of the selective decay principles still lacks rigorous proof, they can and have been applied in a number of areas of physics, such as MHD and geophysics. Selective decay hypotheses have been supported by a number of computational studies (e.g., Refs. [12,13]) and experimental studies (e.g., successful prediction of the magnetic configuration of reversed field pinch plasmas). Thus, our model based on the minimum enstrophy principle is plausible, and the results are believable and useful. The minimum enstrophy state is a subclass of stable states. When there is no external forcing and dissipation, the minimum enstrophy state is one of the possible attractors. In the presence of viscous damping, the minimum enstrophy state is the attractor of the system. However, when the viscosity approaches zero, the system may be trapped in long-lived quasistationary states while relaxing to equilibrium, like many other long-range interacting systems. Thus, the time scale of convergence needs to be considered carefully to determine the relevancy of the minimum enstrophy model to inertial time scales.

Turbulence spreading [14–16] is related to PV mixing because the transport of turbulence intensity has influence on Reynolds stresses and flow dynamics. The momentum theorems for the zonal flow in Rossby or drift wave turbulence [17] link turbulent flux of potential enstrophy density to zonal flow momentum and turbulence pseudomomentum, along with the driving flux and dissipation. In this work, turbulence spreading is linked to PV mixing via the relation of energy flux to PV flux. The turbulent flux of kinetic energy density during minimum enstrophy relaxation is shown to be proportional to the gradient of the (ultimately homogenized) quantity, which is the ratio of PV gradient to the zonal flow. A possible explanation of up-gradient transport of PV due to turbulence spreading, based on the connection between PV mixing and turbulence spreading, is discussed in the last section.

II. DEDUCING THE FORM OF THE PV FLUX

We approach the question of the dynamics of momentum transport in 2D turbulence by asking what the form of PV flux must be to dissipate enstrophy but conserve energy. We start with the conservative PV evolution equation

$$\partial_t q + v \cdot \nabla q = \nu_0 \nabla^2 q, \tag{1}$$

where ν_0 is molecular viscosity. Equation (1) states PV as a material invariant and so applies to many quasi-2D systems, including, but not limited to, the following two systems. In 2D quasigeostrophic turbulence [2], the PV and velocity fields are $q = \nabla^2\psi + \beta y$ and $(v_x, v_y) = (-\partial\psi/\partial y, \partial\psi/\partial x)$, where ψ is the stream function and β is the latitudinal gradient of the Coriolis parameter. In drift wave turbulence [18], the PV consists of the ion vorticity due to $E \times B$ drift and the ion density n. In this paper we use the coordinates of a 2D geostrophic system: the x axis is in the zonal direction, the direction of symmetry (the poloidal direction in tokamaks), and the y axis is in the meridional direction, the direction of anisotropy (the radial direction in tokamaks). Periodic boundary conditions in the $\hat{x}$ direction are imposed, and we assume zero mean zonal flow at $\pm y_0$ boundaries and zero PV flux and energy flux through $\pm y_0$ boundaries. We average Eq. (1) over the zonal direction to obtain the mean field equation for PV:

$$\partial_t \langle q \rangle = -\partial_y \Gamma_q + \nu_0 \partial_y^2 \langle q \rangle, \tag{2}$$

where Γ_q is the PV flux in the $\hat{y}$ direction. The selective decay hypothesis states that 2D turbulence relaxes to a minimum enstrophy state. During relaxation, the enstrophy forward cascades to ever smaller scales until it is dissipated by viscosity. Thus the total potential enstrophy

$$\Omega = \frac{1}{2}\int q^2\, dx\, dy \tag{3}$$

must decrease with time. On the other hand, the kinetic energy inverse cascades to large scales and sees negligible or weak coupling to viscous dissipation, as compared with enstrophy. Only frictional drag can damp flow energy at large scales. The rate of large-scale energy drag is much slower than the rate of small scale enstrophy dissipation. Thus the total kinetic energy

$$E = \frac{1}{2}\int (\nabla\psi)^2\, dx\, dy \tag{4}$$

should remain invariant on the characteristic enstrophy dissipation time. Note that only the kinetic energy is conserved in the minimum enstrophy hypothesis, because the nonadiabatic internal energy (i.e., $\sim \langle (\tilde{n}/n - e\tilde{\phi}/T)^2 \rangle$ for drift wave turbulence) forward cascades to dissipation [19]. To ensure that the total kinetic energy is conserved (apart from feeble collisional dissipation) in mean field theory,

$$\begin{aligned}\partial_t E &= -\int \langle\psi\rangle \partial_t \langle q\rangle\, dx\, dy = -\int \partial_y\langle\psi\rangle \Gamma_q \\ &= -\int \partial_y \Gamma_E\, dx\, dy,\end{aligned} \tag{5}$$

the PV flux is necessarily tied to the energy density flux by

$$\Gamma_q = (\partial_y\langle\psi\rangle)^{-1}\partial_y \Gamma_E, \tag{6}$$

where the energy density flux Γ_E is defined as $\langle v_y \frac{(\nabla\psi)^2}{2}\rangle$. The connection between PV flux and energy density flux has a direct implication for turbulence spreading, which we discuss later in this paper. The form of the energy density flux is constrained by the requirement of decay of total potential enstrophy, i.e., by the demand that

$$\partial_t \Omega = -\int \langle q\rangle \partial_y \Gamma_q = -\int \partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle]\Gamma_E < 0. \tag{7}$$

Note that a finite flux at the boundary would contribute a surface integral term to the total enstrophy evolution. PV relaxation at the point y would then become dependent explicitly upon fluxes at boundary, thus rendering the mean field theory manifestly nonlocal. The simplest solution for Γ_E is for it to be directly proportional to $\partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle]$:

$$\Gamma_E = \mu\partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle], \tag{8}$$

where μ is a positive proportionality parameter. The necessary dependence on turbulence intensity is contained in μ. In this mean field theory, μ is not determined. Note that any combination of an odd derivative of $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$ and an even power of $\langle v_x\rangle$, $\langle q\rangle$, or $\partial_y\langle q\rangle$ will contribute a term which dissipates enstrophy. Thus, the solution we present here is the smoothest (i.e., the dominant one in the long wavelength limit) and lowest order (i.e., not combined with any higher power of $\langle v_x\rangle^2$, $\langle q\rangle^2$, or $(\partial_y\langle q\rangle)^2$). The reasons we study the simplest solution are the following: (1) The smoothest solution reveals the leading behavior of the PV flux on the large scale. This is relevant to our concern with the large-scale flow dynamics. The higher order derivatives should be included to study the relaxation dynamics at smaller scales and the finer scale structure of the shear flow. (2) The dependence on the higher powers of the shear flow intensity can be absorbed into μ. The PV flux is then given by the simplest, leading form of Γ_E:

$$\Gamma_q = (\partial_y\langle\psi\rangle)^{-1}\partial_y\{\mu\partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle]\}. \tag{9}$$

The system evolves to the relaxed state, $\partial_t\langle q\rangle = 0$, when $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$ approaches a constant, where the mean PV flux vanishes and the nonlinear term is annihilated, i.e., $q = q(\psi) = \lambda\psi$ annihilates $v\cdot\nabla q$ for λ constant, so $\partial_y q = \lambda\partial_y\psi$ and $\partial_y[(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle] = 0$.

The structure of the PV flux in equation (9) contains both hyperdiffusive and diffusive terms. The mean PV evolution,

$$\partial_t\langle q\rangle = -\partial_y\left\{\frac{1}{\partial_y\langle\psi\rangle}\partial_y\left[\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\psi\rangle}\right)\right]\right\} + \nu_0\partial_y^2\langle q\rangle, \tag{10}$$

shows that hyperviscosity is the leading high k_y dependence, and so it controls the smaller scales. From Eq. (10) we can also prove that hyperviscosity term damps the energy of the mean zonal flow. Therefore, the hyperviscosity represents the nonlinear saturation mechanism for zonal flow growth and partially defines the scale dependence of turbulent momentum flux. The other important implication of Eq. (9) is that the PV flux is explicitly zonal flow dependent. The zonal velocity appears in the denominators of hyperviscosity and viscosity terms, as well as the diffusion coefficient; this is not seen in perturbative analyses (e.g., Refs. [7,8]). We emphasize that within the mean field approach, the selective decay analysis for the PV flux in this work is entirely nonperturbative and contains no assumption about turbulence magnitude.

The prediction of the homogenization of $(\partial_y\langle\psi\rangle)^{-1}\partial_y\langle q\rangle$ in minimum enstrophy relaxation is a new result. It states explicitly that the local zonal flow speed tracks the local PV gradient in the relaxed state, i.e., strong zonal flows are localized to the regions of larger PV gradient. This trend is observed in the PV staircase, in that strong jets produced by inhomogeneous PV mixing peak at PV jump discontinuities [20]. The jetlike pattern of the E × B staircase is also observed in plasma simulations [21]. It is already known from PV invertibility that local zonal flow speed tracks the local PV gradient. However, the theory predicts this behavior without assuming how PV is mixed and what the initial or final PV profile is like. Thus, we show that a relaxed state of flow-tracking-PV gradient results from PV mixing subject to only the selective decay of enstrophy. The theory does not predict that the staircase is an attractor for the system. Figure 1 shows a cartoon of the PV staircase. Strong zonal flows are located around the edges of PV steps. Since $\partial_y\langle q\rangle/\langle v_x\rangle$ is a constant, we can write $\partial_y\langle q\rangle = \sum_i a_i f(y-y_i)$ and $\langle v_x\rangle = \sum_i b_i f(y-y_i)$, where a_i are constants and $b_i = -\lambda a_i$. While the prediction of the detailed form of the function $f(y-y_i)$ is beyond the scope of this work, the constant proportionality between a_i and b_i reconciles the staircase-like, highly structured profiles with

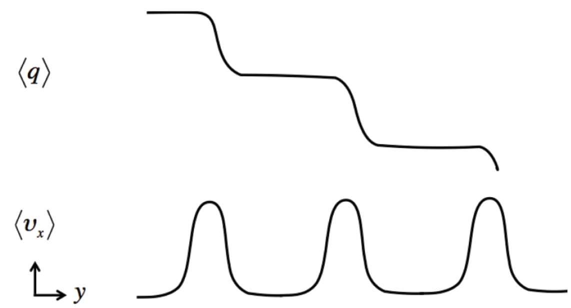

FIG. 1. PV staircase.

the homogenization or mixing process required to produce it. In a related vein, both $\partial_y\langle q\rangle$ and $\partial_y\langle\psi\rangle$ can each be large and variable, though the ratio is constrained.

PV mixing in minimum enstrophy relaxation is also related to turbulence spreading [14–16], since we can see from Eq. (6) that Γ_E and Γ_q are related. Since there is no mean flow in the direction of inhomogeneity, Γ_E represents the effective spreading flux of turbulence kinetic energy and is given by

$$\Gamma_E = -\int\Gamma_q\langle v_x\rangle\,dy = \mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\langle v_x\rangle}\right). \tag{11}$$

Equation (11) shows that $\partial_y\langle q\rangle/\langle v_x\rangle$ drives spreading and that the spreading flux vanishes when $\partial_y\langle q\rangle/\langle v_x\rangle$ is homogenized. The dependence of Γ_E on zonal flow follows from the fact that turbulence spreading is a mesoscale transport process. Note that the step size of the PV staircase, which corresponds to the distance between zonal flow layers, is also mesoscale. Both observations suggest that the relaxation process is a nonlocal phenomena. This is a necessary consequence of PV inversion, i.e., the relation $\nabla^2\psi + \beta y = q$, so that $\langle v_x\rangle$ is an integral of the $q(y)$ profile. Thus $\Gamma_E(y)$ and $\Gamma_q(y)$ in fact depend nonlocally on $q(y)$.

An expression for the relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state. We write $\langle q\rangle = q_m(y) + \delta q(y,t), \langle\psi\rangle = \psi_m(y) + \delta\psi(y,t)$ and use the homogenization condition in relaxed state $\partial_y q_m = \lambda\partial_y\psi_m$. Assuming $\delta q(y,t) = \delta q_0\exp(-\gamma_{\rm rel}t - i\omega t + iky)$, the relaxation rate is found to be

$$\begin{aligned}\gamma_{\rm rel} &= \mu\left[\frac{k^4+4\lambda k^2+3\lambda^2}{\langle v_x\rangle^2} - \frac{8q_m^2(k^2+\lambda)}{\langle v_x\rangle^4}\right],\\ \omega &= \mu\left(-\frac{4q_m k^3+10q_m k\lambda}{\langle v_x\rangle^3} + \frac{8q_m^3 k}{\langle v_x\rangle^5}\right).\end{aligned} \tag{12}$$

The condition of relaxation, i.e., that modes are damped, requires positive $\gamma_{\rm rel}$: $k^2 > 8q_m^2/\langle v_x\rangle^2 - 3\lambda$, and so $k^2 > 0$ relates q_m to λ and $\langle v_x\rangle$ by

$$\frac{8q_m^2}{\langle v_x\rangle^2} > 3\lambda. \tag{13}$$

Equation (13) shows that the zonal flow cannot grow arbitrarily large and is constrained by the potential enstrophy density and scale parameter λ. It also shows that a critical residual enstrophy density q_m^2 is needed in the minimum enstrophy state, so as to sustain a zonal flow of a certain level. Equation (13) thus specifies the minimum enstrophy of relaxation. Therefore, we not only obtain the structure of the end state, which is expressed in terms of λ, the constant of proportionality between PV gradient and zonal flow velocity, but also we observe that potential enstrophy intensity and zonal flow strength are ultimately related in the relaxed state.

One can define a characteristic scale:

$$l_c = \left| \frac{\partial_y \langle q \rangle}{\langle v_x \rangle} \right|^{-1/2}. \tag{14}$$

In minimum enstrophy state, $l_c = |\lambda|^{-1/2}$ and PV flux can vanish on scale l_c. As a result, l_c characterizes the scale at which the terms in the PV flux can compete and cancel. For scales smaller than l_c, hyperviscosity dominates and damping wins. For scales larger than l_c, effective viscosity (which can be negative) dominates. It is interesting to compare l_c with the Rhines scale [22] $l_R \sim (\partial_y \langle q \rangle / \tilde{v}_{\rm rms})^{-1/2}$, where $\tilde{v}_{\rm rms}$ is the r.m.s. velocity at the energy-containing scales. Which velocity should really be used to calculate the Rhines Scale is still being debated (see, e.g., Refs. [23] and [24]). l_c and l_R both depend on the gradient of mean field PV; what distinguishes them is that l_c is determined by mean zonal velocity while l_R is set by fluctuation velocity. The characteristic scale and Rhines scale become indistinguishable when $\tilde{v}_{\rm rms}$ reaches the level of zonal flow velocity.

III. DISCUSSION AND SUMMARY

In this paper, we have considered the problem of zonal flow formation in quasi-2D turbulent systems which conserve PV. The approach is to study PV transport during relaxation processes by exploiting the minimum enstrophy relaxation principle. The analysis of PV flux using selective decay is nonperturbative and so can be applied to general 2D turbulent systems. The nonlinear term is annihilated in the end state of the selective decay. The deduced PV flux is shown to be non-Fickian; it consists of diffusive and hyperdiffusive terms. Note that there are other forms of PV flux which can minimize enstrophy while conserving energy. In this work, we study the simplest, smoothest form of the PV flux. The hyperviscosity reflects the saturation mechanism of zonal flows and the scale dependence of the momentum flux. The results are pragmatically useful in the context of transport modeling, where the problems of zonal flow scale and saturation are important. The homogenized quantity in the relaxed state is found to be the ratio of PV gradient to zonal flow velocity, implying that strong localized zonal flows are located at sharp PV gradients. This is consistent with the structure of the PV staircase. A relaxation rate is derived using linear perturbation theory. We show that a critical enstrophy in the minimum enstrophy state is needed to sustain zonal flows at a given level. A characteristic scale l_c is defined from the homogenized quantity, $l_c = |\partial_y \langle q \rangle / \langle v_x \rangle|^{-1/2}$, so that hyperviscosity dominates at scales smaller than l_c. l_c is similar to the Rhines scale.

We compare our model with previous relaxation models for geostrophic turbulent flow (e.g., Refs. [25–27]). The main difference is that Eq. (10) is derived using a structural approach, while the previous relaxation equations are derived using variational principles, with various conserved and dissipating or maximizing functionals. Our result from a structural approach is consistent with the result from the calculus of variations [6], in which the enstrophy is minimized at constant energy, so $\delta\Omega + \lambda\delta E = \int q \delta(\nabla^2\psi)\, dx\, dy + \lambda \int \nabla\psi \cdot \nabla\delta\psi\, dx\, dy = \int (q - \lambda\psi)\nabla^2\delta\psi\, dx\, dy$ is required to vanish, and so $\langle q \rangle \langle \psi \rangle^{-1}$ is equal to the Lagrange multiplier λ. What we show here is that our structural approach also gives $\langle q \rangle \langle \psi \rangle^{-1} = \text{const}$. First, instead of writing the nonlinear term in PV equation as an explicit divergence of a PV flux, we keep it as N and repeat the minimum enstrophy analysis as we did in the paper. Conservation of the total kinetic energy in mean field theory gives $\partial_t E = -\int \langle \psi \rangle \langle N \rangle\, dx\, dy = -\int \partial_y \Gamma_E\, dx\, dy$. Thus, the nonlinear term is necessarily tied to the energy flux by $\langle N \rangle = \langle \psi \rangle^{-1} \partial_y \Gamma_E$. The form of the energy density flux is constrained by the requirement of decay of total potential enstrophy: $\partial_t \Omega = \int \langle q \rangle \langle N \rangle\, dx\, dy = \int \langle q \rangle \langle \psi \rangle^{-1} \partial_y \Gamma_E\, dx\, dy < 0$, which in turn forces $\Gamma_E = \nu \partial_y (\langle q \rangle \langle \psi \rangle^{-1})$. The system evolves to the relaxed state when $\partial_y (\langle q \rangle \langle \psi \rangle^{-1}) = 0$. Therefore, the structural approach we use in this paper can recover the condition of $\langle q \rangle \langle \psi \rangle^{-1} = \text{constant}$ in the steady state.

In the paper, we write the form of the nonlinear term as an explicit divergence of a PV flux, i.e., we take $\langle N \rangle = -\partial_y \Gamma_q$. The difference between the results of the N and the $\partial_y \Gamma_q$ formulations comes from the treatments of the structure of the nonlinear term. We can see clearly how the treatment of derivatives results in difference forms of the homogenized quantities in the two approaches: $\langle \psi \rangle^{-1} \langle q \rangle$ and $(\partial_y \langle \psi \rangle)^{-1} \partial_y \langle q \rangle$. The derivative of equation $\langle q \rangle = \lambda \langle \psi \rangle$, from the N approach, gives the equation $\partial_y \langle q \rangle = \lambda \partial_y \langle \psi \rangle$, obtained from the Γ_q approach. Thus, the two solutions are consistent with each other, and are both consistent with the solution from the calculus of variations. The $\partial_y \Gamma_q$ formulation is more accurate, since it starts with a more precise form of the nonlinear term in PV equation, i.e., to take N as a divergence of a PV flux. Γ_q is smoother than N, and hence better satisfies the conditions of the mean-field approximation, namely, that the fluctuations around the average value be small, so that terms quadratic in the fluctuations can be neglected. Moreover, while the stream function ψ is unique up to an arbitrary constant, the absolute value of its derivative $\partial_y \psi = -v_x$ has a clear physical meaning. Therefore, in this paper we maintain the form of the nonlinear term in the mean PV evolution as an explicit divergence of a PV flux.

We note that even though the result can give a staircase-like relationship between the zonal flow and PV profiles in the relaxed state, the solution is not suitable to explain the sharp jump in the PV profile of the staircase. Equation (8) is the solution of the form of energy flux which dominates the large scale. This solution can represent two-scale phenomena but cannot treat multiscale phenomena. We also note that the model presented in this paper does not directly predict staircases. It does, however, predict a relaxed state with a structure which is consistent with PV staircases, namely, the proportionality between mean PV gradient and zonal flow strength. The model also shows that a system with flow structure similar to that of staircases arises as a consequence of PV mixing during the minimum enstrophy relaxation. The form of PV flux is shown to contain not only diffusive but also hyperdiffusive transport of PV. (Note that simple PV diffusion cannot recover a structure consistent with PV staircases.) These results are not seen in the previous relaxation models. Thus, the model provides a new way to look at the problem of staircase formation from the perspective of turbulent relaxation. The staircase formation will depend on the initial conditions. Even though in this model we do not derive the evolution and end state of a given initial state, we show that the mean field PV will evolve toward a state, at which PV and energy fluxes vanish.

PV mixing, the fundamental process for zonal flow generation, in wave-number space is directly linked to the forward enstrophy cascade. The importance of such small-scale mixing processes is seen from the appearance of hyperviscosity in the PV flux, which contributes to zonal flow energy damping. The terms in the PV flux which contribute to zonal flow energy growth (i.e., effective negative viscosity), however, are not well reconciled with the picture of diffusive mixing of PV in real space. Here we offer a possible explanation, based on the connection between PV mixing and turbulence spreading derived from the minimum enstrophy analysis, i.e., $\Gamma_E = -\int \Gamma_q \langle v_x \rangle \, dy \sim \nabla(\langle q \rangle'/\langle v_x \rangle)$. We may consider turbulence spreading as a process which contributes to up-gradient, or antidiffusive, mixing of PV. The argument is as follows: It is reasonable to assume that PV mixing in real space tends to transport PV from the region of larger mean PV to the region of smaller mean PV. Because a stronger mean vorticity corresponds to a stronger shearing field which suppresses turbulence, the PV mixing process tends to transport PV away from the region of weak excitation toward the region of stronger excitation. In contrast, the spreading of turbulent enstrophy tends to transport enstrophy from the strongly turbulent region to the weakly turbulent region. When the tendency of turbulence spreading is greater, the net transport of PV appears up-gradient, and so the apparent effective viscosity becomes negative. The relaxed state is reached when PV mixing and turbulent enstrophy spreading are balanced. The total PV flux that we calculate in the relaxation model includes both trends.

We conclude by noting that, the dynamics of PV flux derived analytically in this work has not been confirmed by numerical tests. Therefore, an important topic for future research would be developing a numerical simulation test and comparing its results with the analytical predictions.

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